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Displacement control crack-growth instability in an elastic-softening material

Part III *General analysis for a small softening zone*

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The paper formulates a general criterion for the displacement control instability of a crack in an elastic-softening solid, with attention being focused on the situation where the softening zone size is small in relation to the solid's characteristic dimension. The instability criterion is expressed in terms of the material's softening behaviour and the solid's geometrical parameters.

1. Introduction

Elastic-softening materials, such as for example concrete, cement and fibre-reinforced brittle ceramics, are characterized by a behaviour such that when a pre-cracked solid is progressively loaded, the material in the vicinity of the crack tip fractures and the crack extends. Behind the propagating crack tip, there is a zone of partially fractured material and the unfractured material elements within this zone exert a restraining stress between the crack faces. The effect of these elements can be averaged to give a restraining stress (p) versus relative displacement (v) behaviour for the crack faces, i.e. a p–v softening law. The stress has a finite value p_c at the crack tip and decreases as the relative displacement increases. When the opening at the trailing edge of the softening zone, i.e. the initial crack tip position, attains a critical value δ_c, the restraining stress p becomes zero and the softening zone is then said to be fully developed.

Carpinteri [1–3] has focused attention on the global response of a cracked elastic-softening solid when it is subjected to displacement control loading, giving particular consideration to the behaviour of an edge-cracked solid that is subjected to three-point bending deformation; he examined [1, 2] the behaviour of a concrete-type material whose fully developed softening zone size was very large. By analysing a range of solid dimensions where the solid width, length and crack depth were scaled proportionally, Carpinteri showed that displacement control crack-growth instability was favoured by large dimensions, and also by a small crack depth/solid width ratio; he referred [1] to experimental results which support the theoretical predictions. In Part I of this series of papers [4], the author has extended Carpinteri's study of the bend configuration to the case where the softening zone is very small in comparison with other characteristic dimensions of the configuration. Indeed the author assumed that the softening zone size was infinitesimally small, and consequently he performed a simple linear elastic analysis, assuming that the crack extension condition could be viewed in terms of the stress intensity factor K being equal to K_c, a measure of the fracture resistance due to the restraining effect of the material within the softening zone. He showed that the criterion for a displacement control crack-growth instability can be expressed in the form $a/W < g(L/W)$ where a = crack depth, W = beam width, L = beam length and $g(L/W)$ is an increasing function of L/W. The criterion is therefore independent of material properties (i.e. K_c), and depends only on geometrical parameters through the ratios a/W and L/W, though not on the magnitudes of the dimensions themselves. This contrasts with Carpinteri's results [1, 2] for a material with a large softening zone, which showed that a displacement control crack-growth instability was favoured by large dimensions. In Part II [5], the author analysed the model of a solid containing two symmetrically situated deep cracks and with tensile loading of the remaining ligament. With this model, for the special case where the stress retains a constant value p_c within the softening zone, the behaviour of materials having large and small softening zones can be considered within the framework of the same relatively simple analytical procedure. The analysis defined the condition for a displacement control crack-growth instability, the condition being expressed in terms of the material's softening zone characteristics (p_c and δ_c) and the solid's geometrical parameters; the results were, in general, consistent with those obtained [1, 2, 4] for the bend configuration.

As part of the continuing effort in this area, this paper develops a small-zone analysis which is applicable to a situation where the softening zone is not infinitesimally small as has been assumed in Part I [4], but instead is a small fraction of a solid's characteristic dimension. The results of the general analysis are applicable to any configuration, and is conducted in the context of the importance of the problem. Many engineering structures are subjected to displacement control loading, and if there is an instability then the load will immediately drop. Though there may be a stability on the lower portion of the load–displacement curve after the load drop, according to the results of a static analysis, the energy associated with the load reduction may well lead to catastrophic dynamic failure of the structure. This possibility is the motivation for research in this particular area of materials engineering.

2. General small-zone analysis

Assume that a solid of thickness B deforms under Mode I plane strain conditions, the solid being subjected to a displacement Δ which generates a load P. If the solid is linear elastic, δ and P are related by an expression of the form

$$\Delta = C_M P \quad (1)$$

where C_M is a compliance furation which is dependent on the solid's geometrical parameters and in particular the crack size a. C_M is related to the stress intensity K due to the applied loading P by the standard relation

$$\frac{dC_M}{da} = \frac{2BK^2}{E_0 P^2} = \frac{2H^2}{E_0 B} \quad (2)$$

if K is expressed in the form $K = HP/B$ where H is a function of the crack size; $E_0 = E/(1 - \upsilon^2)$ where E is Young's modulus and υ is Poisson's ratio.

Now as the loading is applied to an elastic-softening solid, the material fractures at the crack tip at an infinitesimally small K value (if the matrix fracture resistance is ignored), and as the loading is progressively increased the crack extends, leaving in its wake a softening zone. It will be assumed that the stress is constant within the softening zone and has a value p_c, this stress being operative until the opening v at the trailing edge of the softening zone, i.e. the initial crack tip position, attains a critical value δ_c when the restraining stress abruptly falls to zero; the softening zone is then said to be fully developed. Thereafter, the crack continues to extend along with a fully developed softening zone, there being a constant opening δ_c at the trailing edge of the softening zone. We are therefore concerned with the instability of a crack growing along with its fully developed softening zone, and to quantify this problem it will be assumed that the crack length a is measured to the trailing edge of the fully developed zone, and not to the leading edge. Since the zone is assumed to be small in comparison with the solid's characteristic dimension, the problem can be considered in terms of an effective crack length, which is given by Irwin's method [6] for modifying the crack length by accounting for the non-linearity within the softening zone. Thus the load-point displacement Δ can now be related to the load P by an expression with the form of Equation 1 but with C_M expressed in terms of an effective crack length, i.e.

$$\Delta = C_M(a_{EFF})P \quad (3)$$

where the effective crack length a_{EFF} is related to the actual crack length by Irwin's relation [6]

$$a_{EFF} = a + a_e$$
$$a_{EFF} = a + \frac{\pi K^2}{24 p_c^2} \quad (4)$$

The second term a_e is the distance from the crack tip to the effective crack tip, and its value [7] (see Equation 4) is appropriate to the Dugdale–Bilby–Cottrell–Swinden model [8, 9] where there is a constant stress (p_c) zone at a crack tip. In Equation 4, K is the crack tip stress intensity determined as if the material is linear elastic, i.e. it is equal to HP/B with H and C_M being related via Equation 2. Now the magnitude of the J integral [10], according to the effective crack procedure, is given by the expression

$$J = J(a_{EFF}) = \frac{1}{E_0}[K(a_{EFF})]^2 \quad (5)$$

which on expansion to the first two terms leads to the result

$$J = \frac{P^2H^2}{E_0B^2} + \frac{\pi P^4 H^3}{12 E_0 p_c^2 B^4}\left(\frac{dH}{da}\right) \quad (6)$$

For a fully developed softening zone where $J = J_{IC} = p_c\delta_c$, it follows that

$$J_{IC} = p_c\delta_c = \frac{P^2H^2}{E_0B^2} + \frac{\pi P^4 H^3}{12 E_0 p_c^2 B^4}\left(\frac{dH}{da}\right) \quad (7)$$

Expansion of Equation 3, again to the first two terms, coupled with the use of Equation 4, gives the relation

$$\Delta = PC_M + \frac{\pi P^3 H^2}{24 p_c^2 B^2}\left(\frac{dC_M}{da}\right) \quad (8)$$

Equation 2 allows Equations 7 and 8 to be written respectively in the forms

$$J_{IC} = \frac{P^2 C_M'}{2B} + \frac{\pi E_0 P^4}{96 p_c^2 B^2} C_M' C_M'' \quad (9)$$

$$\Delta = PC_M + \frac{\pi E_0 P^3}{48 p_c^2 B}(C_M')^2 \quad (10)$$

where the primes refer to derivatives with respect to crack length. Differentiating Equations 9 and 10 gives the expressions

$$0 = \frac{1}{2B}(2PC_M'\delta P + P^2 C_M''\delta a) + \frac{\pi E_0}{96 p_c^2 B^2}\{4P^3 C_M' C_M''\delta P + P^4[(C_M'')^2 + C_M' C_M''']\delta a\} \quad (11)$$

$$\delta\Delta = C_M\delta P + PC_M'\delta a + \frac{\pi E_0}{48 p_c^2 B}[3P^2(C_M')^2\delta P + 2P^3 C_M' C_M''\delta a] \quad (12)$$

Equations 11 and 12, by elimination of δa, show that to the first two terms in P

$$\frac{\delta\Delta}{\delta P} = C_M - \frac{2(C_M')^2}{C_M''} - \frac{\pi E_0 P^2 (C_M')^2}{16 p_c^2 B}\left(1 - \frac{2C_M' C_M'''}{3(C_M'')^2}\right) \quad (13)$$

For there to be a displacement control crack-growth instability upon the attainment of a fully developed softening zone, $d\Delta/dP$ should be negative. Equation 13 thus shows that the instability criterion (to the first two terms in P) is

$$C_M - \frac{2(C'_M)^2}{C''_M} - \frac{\pi E_0 P^2 (C'_M)^2}{16 p_c^2 B}\left(1 - \frac{2C'_M C'''_M}{3(C''_M)^2}\right) > 0 \tag{14}$$

Since Equation 9 shows that $P^2 C'_M/2B$ is equal to $J_{IC} = p_c \delta_c$ as a first approximation, it follows for a positive geometry where the function H increases with crack length (see Equation 2) that the instability criterion (Relation 14) can be written in the form

$$C_M - \frac{2(C'_M)^2}{C''_M} - \frac{\pi E_0 \delta_c C'_M}{8 p_c}\left(1 - \frac{2C'_M C'''_M}{3(C''_M)^2}\right) > 0 \tag{15}$$

and since the effective size ($a_{e\infty}$) of a fully developed softening zone associated with a semi-infinite crack in a remotely loaded infinite solid is given by the expression [7]

$$a_{e\infty} = \frac{\pi E_0 \delta_c}{24 p_c} \tag{16}$$

the instability criterion can be written in the form

$$C_M - \frac{2(C'_M)^2}{C''_M} - a_{e\infty}\left(3 - \frac{2C'_M C'''_M}{(C''_M)^2}\right) > 0 \tag{17}$$

Inspection of this relation shows how the softening behaviour of the material (i.e. the magnitude of the parameter $a_{e\infty}$) enters into the instability criterion with a small-zone analysis. Of course, in the special situation where the softening zone is infinitesimally small, the second term on the left-hand side of the inequality (Relation 17) disappears and the instability criterion then simplifies to that obtained via the analysis in Part I [4], i.e.

$$C_M - \frac{2(C'_M)^2}{C''_M} > 0 \tag{18}$$

and in this case, as indicated in section 1, the instability criterion does not involve the material properties, but only the geometrical parameters of the solid.

Before concluding this section, it should perhaps be pointed out that the preceding analysis has been with regard to a situation where there is a single crack tip, as for example with the compact tension specimen or bend specimen geometrical configuration. For a symmetric loading situation where extension of a crack of length $2a$ occurs simultaneously at two tips, as for example with the centre-cracked tension or double edge notch tension configuration, though the factor 2 in Equation 2 is replaced by a factor 4, it is readily seen that the instability criterion is still of the form of Relation 17, but with the primes now referring to derivatives with respect to the half crack length a.

3. Special case of double edge notch tension configuration

Consider the model (Fig. 1) of a solid of width $2h$, height D and thickness B in the direction of the figure

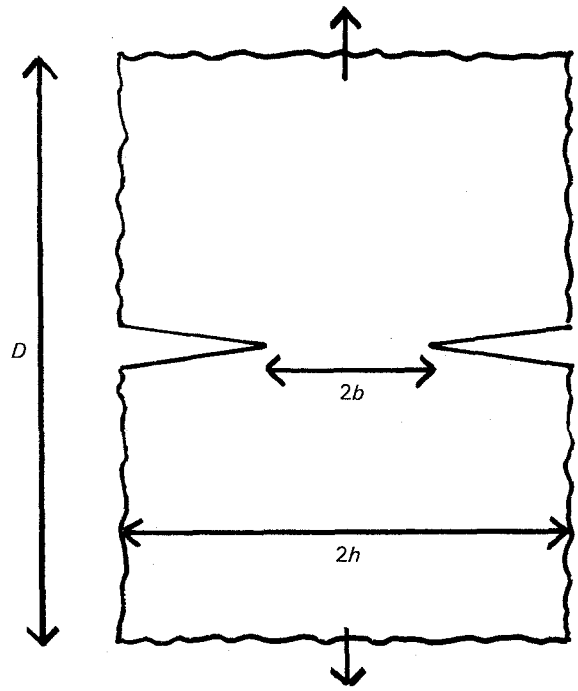

Figure 1 Model of the double edge notch tension configuration.

normal. The solid contains two symmetrically situated deep cracks, and is subjected to a displacement Δ at a point along the axis which bisects the ligament which is of initial width $2b$, this displacement being associated with a load P. Softening zones develop at each crack tip and the concern is with regard to the criterion for a displacement control crack-growth instability when the zones are fully developed, for the case where the fully developed softening zone size is small in comparison with the solid's characteristic dimension, i.e. b which is $\ll h$ and is $\ll D$. For this configuration, the compliance C_M relating Δ and P is

$$C_M = \left[\frac{D}{2h} + \frac{4}{\pi}\ln\left(\frac{2h}{\pi b}\right)\right]\frac{1}{BE_0} \tag{19}$$

and it immediately follows from Relations 17 and 19 that the instability criterion is

$$\frac{D}{2h} + \frac{4}{\pi}\ln\left(\frac{2h}{\pi b}\right) - \frac{8}{\pi} + \frac{4a_{e\infty}}{\pi b} > 0 \tag{20}$$

with $a_{e\infty}$ being given by Equation 16.

Now, as indicated in section 1, this particular model has been analysed in Part II [5] for the general case where the softening zone size is not necessarily small. It was there shown that the criterion for a displacement control crack-growth instability associated with the full development of a softening zone is

$$\frac{\pi D}{4h} + 2\ln\left(\frac{2h}{\pi b}\right) > g(\lambda) = \frac{4\ln(1-\lambda)\ln(1+\lambda)}{\ln(1-\lambda) + \ln(1+\lambda)} \tag{21}$$

where $\lambda = P/2bBp_c$ and with the condition for the full development of a softening zone being

$$\delta_c = \frac{4p_c b}{\pi E_0}[(1+\lambda)\ln(1+\lambda) + (1-\lambda)\ln(1-\lambda)] \tag{22}$$

Equations 21 and 22 are applicable to the general situation where a softening zone is fully developed prior to the softening zones completely traversing the ligament, i.e. for all $\lambda < 1$. However, for the case where the softening zone size is small in comparison with the ligament width, Equations 21 and 22 show that the displacement control crack-growth instability criterion simplifies to Relation 20. There is therefore accord between this paper's general small-zone theory predictions and those arising from the specific analysis in Part II [5].

4. Special case of bending of a small ligament

Consider the model (Fig. 2) of a rectangular beam of length L, width W and thickness B in the direction of the figure normal, containing an edge crack with depth a at the beam mid-section; $b = W - a$ is the remaining ligament width. The ends of the beam are subjected to a relative rotation θ which is associated with a moment M. For the special case where the remaining ligament width b is small in comparison with W and L, the relation between θ and M for the linear elastic situation is

$$\theta = C_M M = \left(\frac{12L}{E_0 B W^3} + \frac{15.8}{E_0 B b^2}\right) M \quad (23)$$

this relation being analogous to Equation 1 for the corresponding applied displacement situation. The small-zone displacement control crack-growth instability criterion is still Relation 17 with C_M being given by Equation 23. It is therefore

$$\frac{L}{W} - 0.44\frac{W^2}{b^2} - \frac{0.88 a_{e\infty}}{b}\left(\frac{W^2}{b^2}\right) > 0 \quad (24)$$

with $a_{e\infty}$ being given by Equation 16. Inspection of this relation again shows how the softening behaviour of the material, through the magnitude of the parameter $a_{e\infty}$, enters into the instability criterion with a small-zone analysis. Of course, for the special situation where the softening zone is infinitesimally small, the last term on the left-hand side of the inequality (Relation 24) disappears and the instability criterion simplifies to that obtained in Part I [4], namely

$$\frac{L}{W} > 0.44\frac{W^2}{b^2} \quad (25)$$

and in this case the instability criterion does not involve the material properties, but only the geometrical parameters of the solid. Interestingly, a comparison of Relations 20 and 24 for respectively the tension and bending situation shows that an increase in the parameter $a_{e\infty}$ is conducive to instability in the tensile case, but is conducive to stability in the bending situation. This difference is a consequence of the sign of the term involving $a_{e\infty}$ in the general instability criterion (Relation 17) being different for the two cases.

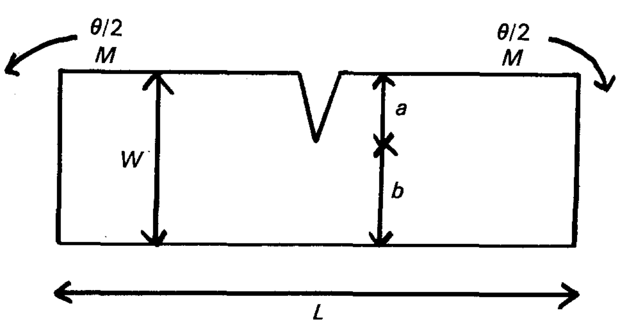

Figure 2 Model of the bend specimen configuration.

5. Special case of centre-cracked tension panel

Consider the model (Fig. 3) of a solid of width $2h$, height D and thickness B in the direction of the figure normal. The solid contains a centrally situated crack of length $2a$, and is subjected to a displacement Δ at a point along the central axis, this displacement being associated with a load P. Softening zones develop at each crack tip and the concern is with regard to the criterion for a displacement control crack-growth instability when the zones are fully developed, for the case where the fully developed softening zone size is small in comparison with the solid's characteristic dimension. It will furthermore be assumed that the crack size $2a$ is small in comparison with both the panel width $2h$ and height D. The model can then be viewed as approximately simulating the behaviour of an edge crack of depth a in a wide tension panel. With the preceding assumptions, the compliance C_M relating Δ and P may be written in the form

$$C_M = \left(\frac{D}{2h} + \frac{\pi a^2}{2h^2}\right)\frac{1}{E_0 B} \quad (26)$$

and it immediately follows from Relations 17 and 26 that the instability criterion is

$$\frac{D}{2h} - \frac{3\pi a^2}{2h^2} - \frac{3\pi a_{e\infty}}{a}\left(\frac{a^2}{h^2}\right) > 0 \quad (27)$$

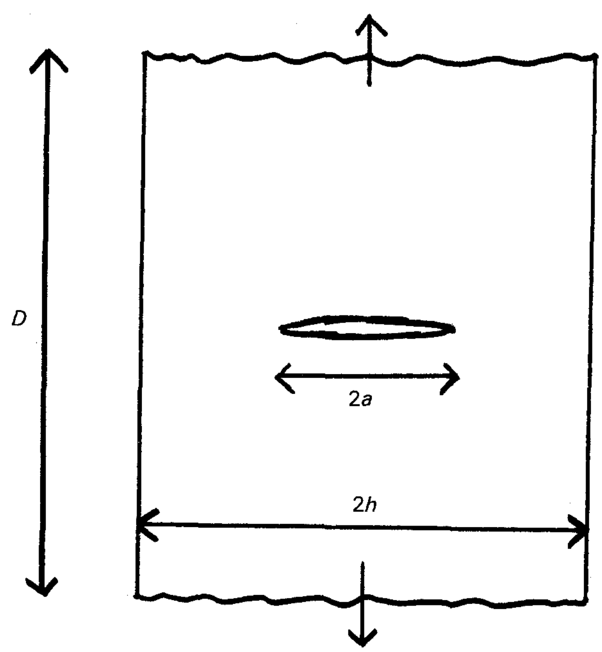

Figure 3 Model of a centre-cracked tension panel.

with $a_{e\infty}$ being given by Equation 16. Comparison of Relations 20 and 27 for respectively the large and small tensile crack situations shows that an increase in the parameter $a_{e\infty}$ is conducive to instability in the large crack case, but is conducive to stability in the small crack case.

6. Discussion

This paper has formulated a general criterion (Relation 17) for the displacement control instability of a crack in an elastic-softening solid for the case where the softening zone size is small in relation to the solid's characteristic dimension. The work is therefore an extension of Part I's earlier analysis [4] which was based on the assumption of linear elastic behaviour with the softening zone regarded as being infinitesimally small. The general instability criterion is expressed in terms of the solid's geometrical parameters through the compliance function C_M appropriate for linear elastic behaviour, and the material's softening behaviour through the way in which this affects the parameter $a_{e\infty}$ – the effective size of fully developed softening zone associated with a semi-infinite crack in a remotely loaded infinite solid. The theory has been developed in the present paper for the idealized situation where the stress within the softening zone retains a constant value p_c until the crack opening attains a critical value δ_c, when the stress in the zone falls abruptly to zero and the softening zone is then fully developed. Though the theory has been developed for this idealized softening behaviour, since we are dealing with the case where the softening zone size is small there is no reason to believe that the instability criterion for a general p–v softening behaviour will be different to that derived in this paper (i.e. Relation 17), except that the $a_{e\infty}$ value will of course be different. $a_{e\infty}$ depends on the p–v softening law [11], being influenced by the maximum stress p_c and the maximum displacement δ_c, and also by the precise p–v variation: a general conclusion [12] is that $a_{e\infty}$ increases above the value given by Equation 16 as the softening becomes pronounced, i.e. when the area under the p–v curve is $\ll p_c\delta_c$.

The general conclusion that emerges from the present paper's small-zone analysis is that the criterion for displacement control crack-growth instability involves both geometrical and material softening parameters, but that these effects are coupled in a complicated manner. Thus an increase in $a_{e\infty}$ is conducive to stability with tensile loading for a small crack but is conducive to instability of a deep crack; on the other hand an increase in $a_{e\infty}$ is conducive to stability for a deep crack with bend loading.

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